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Forward

The present booklet contains test problems and solutions of five contests. Every year during the month of May a contest (International Mathematical Olympiad Selection Contest) is held and around five hundred secondary school students participate in the contest. Since 1987 the contests have been organized by the International Mathematical Olympiad Hong Kong Committee (IMOHKC). A purpose of the contests is to select students for further training so that they may participate in national and international mathematical competitions. A student may also learn something about problem solving and consider participating in a contest as an interesting extracurricular activity. The contests included in this booklet were conducted during May of the years 1997-2001. Problems of the contests were set by members of IMOHKC.

Comments and suggestions concerning this booklet may be sent to:

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Contest 97-98

1. Let a, b, c be real numbers such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2 \qquad \text{and} \qquad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$$

Find the value of $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$. (1 mark)

2. On a Cartesian plane every point with integral coordinates is called a lattice point. Let P_n be the lattice point (n, n + 3) and f(n) be the number of the lattice points (excluding the two endpoints) on the line segment joining the origin O and the point P_n. Find f(1) + f(2) + f(3) + …+ f(1997). (1 mark)

3. A positive integer is said to be "good" if the digits of the integer can be divided into two groups such that the sum of the digits in one group is equal to the sum in the other. Find the smallest positive integer n such that n and n + 1 are "good". (1 mark)

4. Let x, y, z be real numbers such that x + y + z = 5 and xy + yz + zx = 3. Find the maximum value of z. (1 mark)

5. Find the number of diagonals that can be drawn in a convex polygon of n (n ≥ 4) sides. (1 mark)

6. Find the time between 1:00 p.m. and 1:30 p.m., correct to the nearest minute, when the hour and minute hand of a clock form an angle of 100^o. (1 mark)

7. In ΔABC, AB = 41 cm, AC = 9 cm and BC = 40 cm. Find the radius of the inscribed circle of ΔABC. (1 mark)

8. Two perpendiculars chords of a circle are at distances a and b respectively from the center. These two chords divide the circle into four pieces. Consider the sum of areas of the largest and the smallest pieces, and the sum of the areas of the other two pieces. Find the difference between these two sums. (1 mark)

9. How many integers from 1 to 1997 have the sum of their digits divisible by 5? (1 mark)

10. The faces of a cube are labeled with different positive integers such that the numbers on any two adjacent faces differ by at least two. Find the minimum value of the sum of all six numbers. (1 mark)

11. A square whose sides are of integral lengths is cut into 25 smaller squares whose sides are also of integral lengths. *Exactly* 24 of these smaller squares are unit squares. Find the area of the original square. (1 mark)

12. Each of the numbers 1, 2, 3, …, 25 is written into a square in a 5×5 table, such that the numbers in each row are in increasing order. Find the maximum value of the sum of the numbers in the third column. (1 mark)

13. Let $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{80}}$. Find the integer n such that $n - 1 < S < n$. (2 marks)

14. A positive integer N (in base 10) is composed of the digits 0 and 1 only, and is divisible by 2475. Find the smallest possible number of digits of N. (2 marks)

15. 20 football teams take part in a tournament. M matches have been played and it is found that
    (a) between any two teams at most one match has been played, and
    (b) among any three teams at least one match has been played between two of them.
    What is the smallest possible value of M. (2 marks)

16. Let a_n be the integer closest to $\sqrt{n}$. Find $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{1997}}$. (2 marks)

17. ABCDEF is a regular hexagon with AB = 1. P and S are the midpoints of AB and ED respectively. The circle with PS as diameter cuts the lines PE and PD at Q and R respectively. Find the area of the quadrilateral QRDE. (2 marks)

18. Points D and E are points inside an equilateral triangle ABC such that DE = 1, DA = EA $= \sqrt{7}$, DB = EC = 2. Find the length of AB. (2 marks)

19. In ΔABC, E, F, G are points on AB, BC, CA respectively such that AE/EB = BF/FC = CG/GA = 1/3. K, L, M are the intersecting points of the lines AF and CE, BG and AF, CE and BG, respectively. Suppose the area of ΔABC is 1, find the area of ΔKLM. (3 marks)

20. Three lines are drawn through a point in a triangle parallel to its sides. The segments intercepted on these lines by the triangle turn out to have the same length. Given the triangle's side lengths a, b and c, find the length of the segments. (3 marks)

Contest 98-99

1. In a sequence $\{a_1, a_2, \ldots, a_n, \ldots\}$, $a_1 = 1$, $4a_na_{n+1} = (a_n + a_{n+1} - 1)^2$, and $a_{n+1} > a_n$. Find a_{1998}. (1 mark)

2. Given that $H_k = \frac{k(k-1)}{2}\cos\frac{k(k-1)\pi}{2}$. Find $H_{19} + H_{20} + \ldots + H_{98}$. (1 mark)

3. A book has 30 chapters. The lengths of the chapters are 1, 2, …, 30 pages. Chapter One starts from page 1 of the book, and each chapter starts from a new page. At most how many chapters can start from an odd-numbered page? (1 mark)

4. On a square carpet of size 123×123, each unit square is colored red or blue. Each red square not lying on the edge of the carpet has exactly five blue squares among its eight neighbors. Each blue square not lying on the edge of the carpet has exactly four red squares among its eight neighbors. Find the numbers of red squares on the carpet. (1 mark)

5. In ΔABC, $\angle C = 90^o$, $\angle A = 30^o$, and $AB = 1$. Let ΔABD, ΔACE, ΔBCF be equilateral triangle with D, E, and F lying outside ΔABC. Let DE intersects AB at G. Find the area of ΔDGF. (1 mark)

6. ABCDEF is a regular hexagon, M, N are points on the segments AC, CE respectively such that $\frac{AM}{AC} = \frac{CN}{CE} = r$. If B, M, N are collinear, find r. (1 mark)

7. ΔABC is an equilateral triangle and P is a variable point on the same plane such that ΔPAB, ΔPBC, and ΔPCA are isosceles triangles. In how many different positions can P lie? (1 mark)

8. The lengths of the three medians AD, BE, and CF of ΔABC are 9, 12, and 15 respectively. Find the area of ΔABC. (1 mark)

9. Find the 4-digit number such that, when the order of its digits is reversed, the new value is 4 time the original one. (1 mark)

10. Determine the number of ordered pairs (x, y), where x and y are integers satisfying the equation $2xy - 5x + y = 55$. (1 mark)

11. Given that [x] represents the greatest integer not exceeding x, find the last three digits of the integer $\left[\dfrac{10^{99}}{10^{33}+3}\right]$. (1 mark)

12. Let $x_0 = 5$ and $x_{n+1} = x_n + \dfrac{1}{x_n}$ for n = 0, 1, 2, … Find the integer closest to x_{1998}.

(2 marks)

13. At least how many of the "+" signs in the expression

$$+1 + 2 + 3 + \ldots + 100$$

must be replaced by "−" signs so that its value is 1998. (2 marks)

14. Except for the first two terms, each term of the sequence 1000, x, 1000 − x, … is obtained by subtracting the preceding term from the one before that. The last term is the first negative term encountered. What positive integer x produces a sequence of maximum length?

(2 marks)

15. In ΔABC, tan A : tan B : tan C = 1 : 2 : 3. Find $\dfrac{AC}{AB}$. (2 marks)

16. Find the smallest positive integer n such that $1997^n - 1$ is divisible by 2^{1998}. (2 marks)

17. In how many ways can 1998 be expressed as the sum of one or more consecutive integers.

(2 marks)

18. What is the largest integer k such that $\dfrac{1001\times 1002\times\cdots\times 1998}{11^k}$ is an integer? (2 marks)

19. Find the smallest multiple of 84 whose digits consist entirely of 6's and 7's only.

(2 marks)

20. An $m\times n\times p$ rectangular box has half the volume of an $(m + 2)\times(n + 2)\times(p + 2)$ rectangular box, where m, n, and p are integers and $m \le n \le p$. What is the largest possible value of p?

(3 marks)

Contest 99-00

1. Let $A_k = \dfrac{19^k + 99^k}{k!}$ for k = 1, 2, 3, …Find k such that A_k is largest. (1 mark)

2. Let x, y, and z be real numbers satisfying $\dfrac{1}{x} + \dfrac{1}{y+z} = \dfrac{1}{2}$, $\dfrac{1}{y} + \dfrac{1}{z+x} = \dfrac{1}{3}$, and $\dfrac{1}{z} + \dfrac{1}{x+y} = \dfrac{1}{4}$. Find x. (1 mark)

3. A solid pyramid VABCD, with a quadrilateral base ABCD, is to be colored on each of the five faces such that no two faces with a common edge will have the same color. If five different colors are available, what is the number of ways to color the pyramid? (1 mark)

4. Let x and y be integers satisfying $y^2 + 3x^2y^2 = 30x^2 + 517$. Find $3x^2y^2$. (1 mark)

5. The digits used to number the pages of a book were counted, and the total numbers of digits used was 2001. Find the number of pages in the book. (1 mark)

6. There exist integer n with the property that n! may be expressed as the product of $n - 3$ consecutive integers. For example $6! = 10\times9\times8$. Find the largest integer satisfying this property. (1 mark)

7. Find the all possible x that satisfy the equation $\left[\dfrac{5+6x}{8}\right] = \dfrac{15x-7}{5}$. Here [u] denotes the greatest integer less than or equal to u. (1 mark)

8. Determine the number of acute-angled triangles with consecutive integer sides and of perimeter not exceeding 100. (1 mark)

9. ABCD is a convex quadrilateral with AD parallel to BC, AC perpendicular to BD, and AC = 5. CF is the perpendicular from C to AD and CF = 4. Find the area of ABCD. (1 mark)

10. A, B, C and D are four vertices of a cyclic quadrilateral, AC and BD meet at E. If BC = CD = 4, AE = 6, and BE and DE are integers, find BD. (1 mark)

11. A sequence of integers $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_n = a_{n-1} - a_{n-2}$ for $n \ge 3$. The sum of the first 1588 terms of the sequence is 1997, and the sum of the first 1997 terms is 1588. Find the sum of the first 1999 terms of the sequence. (2 marks)

12. Find the integer n satisfying $\left[\frac{n}{1!}\right]+\left[\frac{n}{2!}\right]+\cdots+\left[\frac{n}{10!}\right]=1999$. Here [x] denotes the greatest integer less than or equal to x. (2 marks)

13. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15}<\frac{n}{n+k}<\frac{7}{13}$? (2 marks)

14. The pages of a book are numbered form 1 to n. A boy adds up all the page numbers and gets the sum of 1922. It is known that by mistake he adds a certain number twice. What is that number? (2 marks)

15. Four consecutive even integers are removed from the sequence of integers 1, 2, …, n, and the average of the remaining numbers is 51.5625. Determine the largest integer removed. (2 marks)

16. The sequence 1, 3, 4, 9, 10, 12, 13, …consists of all those positive integers, listed in increasing order, which are powers of 3 or sums of distinct powers of 3. Find the 200th term of the sequence. (2 marks)

17. In ΔABC, AB = AC = 12, P is a point on BC such that AP = 8. Find $PB \times PC$. (2 marks)

18. Let ABCD be a rhombus with AB = 5. Suppose $AC \ge 6 \ge BD$, determine the maximum value of AC + BD. (2 marks)

19. A square ABCD of side 12 is divided into 144 unit squares. A circle of radius 6 is drawn touching all the four sides of ABCD. Find the number of unit squares which lie completely within the circle. (2 marks)

20. Determine the minimum value of k, so that for any k numbers chosen from 1, 2, …, 91, there exist two of them, p and q, such that $\frac{2}{3} \le \frac{p}{q} \le \frac{3}{2}$. (2 marks)

Contest 00-01

1. Find the sum of all real x satisfying $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$. (1 mark)

2. In how many ways can 30! be expressed as the product of two positive integers p and q such that $0 < \frac{p}{q} < 1$ and p and q are relatively prime. (1 mark)

3. Find the coefficient of x^{17} in the expansion of $(1 + x^5 + x^7)^{20}$. (1 mark)

4. If [x] represents the greatest integer less than or equal to x, find the sum of

$$\left[\frac{1\times 1999}{2001}\right]+\left[\frac{2\times 1999}{2001}\right]+\left[\frac{3\times 1999}{2001}\right]+\cdots+\left[\frac{2000\times 1999}{2001}\right].$$ (1 mark)

5. If x, y are nonzero numbers satisfying $x^2 + xy + y^2 = 0$. Find the value of

$$\left(\frac{x}{x+y}\right)^{2001}+\left(\frac{y}{x+y}\right)^{2001}.$$ (1 mark)

6. For how many real numbers a do the quadratic equations $x^2 + ax + 8a = 0$ have only integral roots? (1 mark)

7. Suppose tan α and tan β are the roots of $x^2 + \pi x + \sqrt{2} = 0$. Evaluate

$$\sin^2(\alpha+\beta)+\pi\sin(\alpha+\beta)\cos(\alpha+\beta)+\sqrt{2}\cos^2(\alpha+\beta).$$ (1 mark)

8. 2000 lamps are controlled by 2000 switches, numbered 1, 2, 3, …, 2000. A click on each switch will either turn the lamp on or off. In the beginning, all the lamps are off. On the first day, all the switches are clicked once. On the second day, all the switches numbered 2 or a multiple of 2 are clicked once. Similarly on the n^{th} day, all the switches numbered n or a multiple of n are clicked once, and so on. How many lamps will be on after the operation on the 2000^{th} day? (1 mark)

9. Point B is in the exterior of the regular n-sided polygon $A_1A_2 \ldots A_n$ and A_1A_2B is an equilateral triangle. Find the largest value of n such that A_n, A_1 and B are consecutive vertices of a regular polygon. (1 mark)

10. There are three parallel lines L_1, L_2 and L_3 on the plane, with L_2 in between. The distance between L_1 and L_2 is 4, and the distance between L_2 and L_3 is 3. A, B and C are points on L_1, L_2 and L_3 respectively, such that ΔABC is an equilateral triangle. Find the area of the triangle. (1 mark)

11. A circle is inscribed in ΔABC. D, E are points on AB and AC respectively, such that DE is parallel to BC and is tangent to the circle. If the perimeter of ΔABC is p, find the maximum length of DE. (1 mark)

12. In ΔABC, BC = 5, AC = 12, AB = 13. D, E are points on AB and AC respectively such that DE divides ΔABC into two parts of equal area. Find the minimum length of DE. (1 mark)

13. D is a point inside ΔABC. PDS, QDT and RDU are lines parallel to BA, CA and CB respectively such that P, Q lie on BC, R, S lie on CA, and T, U lie on AB. If the areas of ΔTUD, ΔPQD and ΔRSD are respectively 8, 128 and 32, find the area of ΔABC. (1 mark)

14. The numbers $x_1, x_2, \ldots, x_{2000}$ are such that $|x_1 - x_2| + |x_2 - x_3| + \ldots + |x_{1999} - x_{2000}| = 2000$. Find the largest value of

$$|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|,$$

where $y_k = \dfrac{x_1 + x_2 + \cdots + x_k}{k}$, for k = 1, 2, …, 2000. (2 marks)

15. There are n distinct points on a plane. Eight different circles $C_1, C_2, \ldots, C_8$ are drawn such that C_1 passes through one of the points, C_2 passes through two of the points, C_3 passes through three of the points, and so on. Find the minimum value of n. (2 marks)

16. Let S denotes a finite sequence of the letters a and b, and f denotes a function defined by f(S) = the new sequence formed by changing all ‘a’s to ‘a, b’ and all ‘b’s to ‘b, a’. For example, f(b, a, a, b) = (b, a, a, b, a, b, b, a), and the number of pairs of consecutive ‘b’s in f(b, a, a, b) is 1. If $f^{(n)}(S)$ denotes $f(f(\ldots(S)\ldots))$ (n times), find the number of pairs of consecutive ‘b’s in $f^{(n)}(a)$. (2 marks)

17. The circumcircle of the isosceles triangle ΔABC has AB as a diameter. There is a circle Γ tangent to BC at its midpoint E and tangent to the minor arc BC at F. If AB = 4, find the length of the tangent from A to Γ. (2 marks)

18. In ΔABC, BC, CA and AB are divided by P, Q and R respectively in the same ratio. AP intersects BQ at X, BQ intersects CR at Y, and CR intersects AP at Z. Each of the areas of ΔARZ, ΔBPX, ΔCQY and ΔXYZ equals 1 cm^2. Find the area of the quadrilateral PCYX.

(3 marks)

19. B is a point on the line segment AC such that AB = 1 and BC = 3. Semicircles Γ_1, Γ_2, Γ_3 are drawn with diameters AC, AB, BC respectively, and all are on the same side of AC. Let E be on Γ_1 such that $EB \perp AC$. Let U on Γ_2, V on Γ_3 be such that UV is a common tangent to Γ_2 and Γ_3. Find the ratio of the area of ΔEUV over the area of ΔEAC. (3 marks)

20. In each of 12 photographs, there are 3 women; the woman in the middle is the mother of the person on her left and is a sister of the person on her right. The women in the middle of the 12 photographs are all different persons. Determine the smallest number of different persons in the photographs. (3 marks)

Contest 01-02

1. If $\underbrace{200120012001\cdots2001}_{n\ 2001\text{'s}}729$ is divisible by 11, find the least possible value of n.

(1 mark)

2. A carpenter sells armchairs, bookcases and cabinets. A person buys 8 armchairs, 11 bookcases and 2 cabinets and pays \$875. Another person buys 3 armchairs, 2 bookcases and 5 cabinets and pays \$343. Find the total cost of 1 armchair, 1 bookcase and 1 cabinet.

(1 mark)

3. For any positive integer n, let F(n) be the rightmost digit of n in base 10 and let $T_n = F(n^2) - F(n)$, find the sum $T_1 + T_2 + T_3 + \ldots + T_{2002}$. (1 mark)

4. Find the last two digits of the integer $2^{2002}(2^{2003} - 1)$ in base 10. (1 mark)

5. Let x, y, z be positive numbers satisfying the equations

$$x^2 + y^2 - z^2 = \sqrt{3}\,xy \quad \text{and} \quad x^2 - y^2 + z^2 = \sqrt{2}\,xz.$$

Find the ratio y : z. (1 mark)

6. Let b be a positive number. It is known that the equation $x^6 - 2bx^3 + b^2 - 100 = 0$ has exactly two real roots whose difference is 2. Find the value of b. (1 mark)

7. In ΔABC, $\angle BAC = 40^o$ and $\angle ABC = 60^o$. D and E are points on sides AC and AB respectively such that $\angle CBD = 40^o$ and $\angle BCE = 70^o$. Let BD intersects CE at F and AF intersects BC at G. Find $\angle GFC$. (1 mark)

8. ABCD is a square with side length 1 unit. E, F, G and H are points on AB, BC, CD and DA respectively such that $AE = BF = CG = DH = \frac{20}{21}$. Find the area of the region bounded by AF, BG, CH and DE. (2 marks)

9. Let x, y be real numbers such that $x^2 + 4y^2 - 4 = 0$. Find the maximum value of

$$x^2 + 2xy + 4y^2 + x + 2y.$$

(2 marks)

10. A certain number of unit cubes are stuck together to form a cuboid. The six faces of the cuboid, none of which is a square, are painted. If x is the number of unit cubes with no face painted, y is the number of unit cubes with exactly 1 face painted and z is the number of unit cubes with exactly 2 faces painted, then $x - y + z = 2002$. Find the volume of the cuboid.

(2 marks)

11. In ΔABC, M and N are two points on BC such that BM < BN, BM = NC = 4 and MN = 3. If $\angle BAM = \angle MAN = \angle NAC$, find the length of AC. (2 marks)

12. Find the number of 10-digit positive integers such that
    (a) each digit is either 1 or 2, and
    (b) there exist two consecutive 1' s. (2 marks)

13. Integers x and y satisfy $5\times10^7 > x > y$. Suppose $x - 2001$ and $y - 2001$ are respectively the squares of two consecutive integers and d is the greatest common divisor (or highest common factor) of x and y. Find x when d is maximum. (2 marks)

14. Let [x] be the greatest integer less than or equal to x. If $|x+1|-1=\dfrac{x-[x]}{|x-1|}$, find the largest possible value of | x |. (2 marks)

15. Find k such that if P, Q, R, S are points on sides AB, BC, CD, DA respectively of a convex quadrilateral ABCD and

$$\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=k<1,$$

then the area of PQRS is 52% of the area of ABCD. (2 marks)

16. (a, b, c, d) and (a′, b′, c′, d′) are said to be distinct if and only if $a \neq a'$ or $b \neq b'$ or $c \neq c'$ or $d \neq d'$. Find the number of distinct (a, b, c, d) such that a, b, c, d are integers, $1 \leq a < b < c < d \leq 30$ and $a + d = b + c$. (2 marks)

17. p is a prime number such that $\dfrac{1}{p}$, in decimal notations, is a recurring decimal with a period of 7 digits. For example, 4649 is a possible value of p because $\dfrac{1}{4649} = 0.\dot{0}00215\dot{1} = 0.00021510002151\ldots$ Find another possible value of p. (2 marks)

18. A triangular shape paper has sides of lengths $\sqrt{2}, \dfrac{4}{3}, \dfrac{\sqrt{10}}{3}$ units. The paper is folded along a line perpendicular to the side of length $\dfrac{4}{3}$. Find the largest possible overlapped area.

(3 marks)

Answer Keys

Contest 97-98

1. $\frac{3}{2}$	2. 1330	3. 549	4. $\frac{1}{3}$
5. $\frac{n(n-3)}{2}$	6. 1.24 p.m.	7. 4	8. 4ab
9. 399	10. 27	11. 49	12. 85
13. 17	14. 20	15. 90	16. $88\frac{17}{45}$
17. $\frac{25\sqrt{3}}{338}$	18. $\frac{5+\sqrt{13}}{2}$	19. $\frac{4}{13}$	
20. $\frac{2abc}{ab+bc+ca}$			

Contest 98-99

1. 1998^2	2. -40	3. 23	4. 6724
5. $\frac{9\sqrt{3}}{32}$	6. $\frac{\sqrt{3}}{3}$	7. 10	8. 72
9. 2178	10. 16	11. 0, 0, 8	12. 63
13. 17	14. 618	15. $\frac{2\sqrt{2}}{3}$	16. 2^{1996}
17. 16	18. 100	19. 76776	20. 130

Contest 99-00

1. 98	2. $\frac{23}{10}$	3. 420	4. 588

5. 703	6. 23	7. $\frac{7}{15}, \frac{4}{5}$	8. 29
9. $\frac{50}{3}$	10. 7	11. –1179	12. 1165
13. 112	14. 31	15. 28	16. 2943
17. 80	18. 14	19. 88	20. 10

Contest 00-01

1. $\frac{7}{2}$	2. 512	3. 3420	4. 1998000
5. –2	6. 8	7. $\sqrt{2}$	8. 44
9. 42	10. $\frac{37\sqrt{3}}{3}$	11. $\frac{p}{8}$	12. $\sqrt{12}$
13. 392	14. 1999	15. 15	16. $\frac{2^{n-1}+(-1)^n}{3}$
17. $2+\sqrt{2}$	18. $1+\sqrt{5}$	19. $\frac{3}{16}$	20. 19

Contest 01-02

1. 8	2. 112	3. 2	4. 28
5. $\sqrt{2}:1$	6. $6\sqrt{3}$	7. 20^{o}	8. $\frac{1}{841}$
9. $6+2\sqrt{2}$	10. 30030	11. 8	12. 880
13. 16026010	14. $\sqrt{5}$	15. $\frac{2}{3}$	16. 1925
17. 239	18. $\frac{4}{15}$		

Contest 97-98 Solutions

1. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)\Rightarrow\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=\frac{3}{2}$.

2. If (x, y) is a lattice point on the (open) segment joining the origin and P_n , then $\frac{y}{x}=\frac{n+3}{n}\Leftrightarrow ny=x(n+3)$. If n and n + 3 have no common factor greater than 1, then n will divides x, contradicting 0 < x < n. If n and n + 3 have a common factor greater than 1, the factor is 3 and only two lattice points will be on the segment. So f(1) + f(2) + f(3) + …+ f(1997) = 2[1997/3] = 1330.

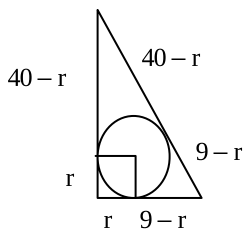
3. Observe that the sum of the digits of good number is even. So n ends in 9. Also n has more than 2 digits because 99 + 1 = 100 is not good. If n = $\overline{xy9}$ (bar means base 10 notation), then n + 1 = $\overline{x(y+1)0}$. So x + y = 9 , x = y + 1 ⇒ x = 5, y = 4 ⇒ n = 549.

4. Since $0\le(x-y)^2=(x+y)^2-4xy=(5-z)^2-4(3-z(5-z))=-3z^2+10z+13=-(z+1)(3z-13)$, we have $-1\le z\le 13/3$. Then max z =13/3 when x = y = 1/3.

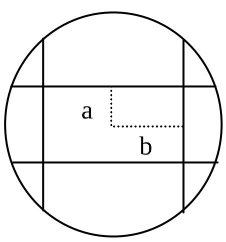
5. There are n – 3 diagonals through each vertex and each diagonal is counted twice. So there are n(n – 3)/2 diagonals.

6. At that moment, let x be the minute, then $100=6x-6(5+\frac{x}{12})$. So $x=23\frac{7}{11}$.
   The time to the nearest minute is 1:24 p.m.

7. Since $9^2+40^2=41^2$, $\angle ACB=90^o$.
   If r is the radius, then
   41 = (40 – r) + (9 – r) ⇒ r = 4.



8. Draw also the mirror images of the chords with respect to the centre. These 4 chords divide the inside of the circle into 9 pieces. The difference between the two sums is the area of the rectangular piece, 4ab.



9. From 1 to 9, there is 1 such integer. From 10n to 10n + 9, where $1 \le n \le 198$, there are two such integers. From 1990 to 1997, there are 2 such integers. So the total is 399.

10. There cannot be three consecutive integers $n - 1, n, n + 1$ (otherwise $n - 1, n + 1$ fall on the face opposite n). So the minimum possible sum is $1 + 2 + 4 + 5 + 7 + 8 = 27$ with 1, 2 on opposite faces, 4, 5 on opposite faces and 7, 8 on opposite faces.

11. Let m, n be the side lengths of the original and the 25th square, respectively. Then $24 = m^2 - n^2 = (m - n)(m + n)$. Since $m - n$, $m + n$ are positive integers and $n \ne 1$, the only possibility is $m = 7$, $n = 5$. Then $m^2 = 49$.

12. Let a(i, j) be the number in the i-th row, j-th column. By permuting the rows, we may assume $a(i, 3) < a(k, 3)$ if $i < k$. Since $a(5, 3) < a(5, 4) < a(5, 5) \le 25$, a(5, 3) is at most 23. Since $a(4, 3) < a(4, 4) < a(4, 5)$ and $a(4, 3) < a(5, 3) < a(5, 4) < a(5, 5) \le 25$, a(4, 3) is at most 20. Similarly a(3, 3), a(2, 3), a(1, 3) are at most 17, 14, 11 respectively. Such a table is possible as shown below. So the maximum is $23 + 20 + 17 + 14 + 11 = 85$.

1	6	11	12	13
2	7	14	15	16
3	8	17	18	19
4	9	20	21	22
5	10	23	24	25

13. Since $\sqrt{k+1} - \sqrt{k} = \dfrac{1}{\sqrt{k+1} + \sqrt{k}} < \dfrac{1}{2\sqrt{k}} < \dfrac{1}{\sqrt{k} + \sqrt{k-1}} = \sqrt{k} - \sqrt{k-1}$, adding these with k = 1, 2, .,.80, we have $8 < \dfrac{1}{2}\left(1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{80}}\right) < \sqrt{80}$. It follows n = 17.

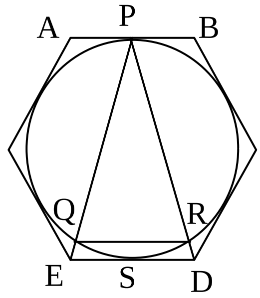
14. Since $2475 = 9 \times 11 \times 25$, N is divisible by 9, 11, 25. Hence the last two digits are both 0, sum of all digits is divisible by 9, sum of digits in the odd and even positions must differ by a multiple of 11. The smallest possible N has 20 digit with eighteen 1's followed by two 0's.

15. Suppose team A played the least number of matches, say k. Each of these k teams played with A must each played at least k matches. This makes a total of at least $k(k + 1)/2$ matches. For the other $(19 - k)$ teams, as they did not play with team A, by (b), every two of them must played with each other, contributing $(19 - k)(18 - k)/2$ matches.

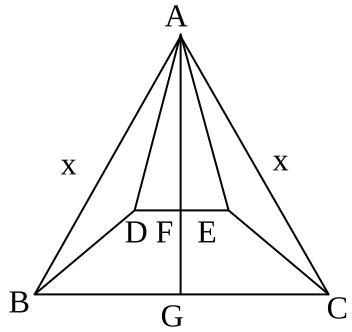
So $M \ge \dfrac{k(k+1)}{2} + \dfrac{(19-k)(18-k)}{2} = k^2 - 18k + 171 = (k - 9)^2 + 90 \ge 90$. So the smallest M is 90 with k = 9.

16. Let k be the nearest integer to $\sqrt{n}$, then $k - \dfrac{1}{2} \le \sqrt{n} < k + \dfrac{1}{2} \Rightarrow k^2 - k + 1 \le n \le k^2 + k$. So $a_n = k$ for $n = k^2 - k + 1$ to $n = k^2 + k$, a total of 2k terms. Then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{1997}}=\frac{1}{1}+\frac{1}{1}+\cdots+\frac{1}{44}+\underbrace{\frac{1}{45}+\cdots+\frac{1}{45}}_{1997-1980=17\text{ terms}}=2\times 44+17\times\frac{1}{45}=88\frac{17}{45}.$$

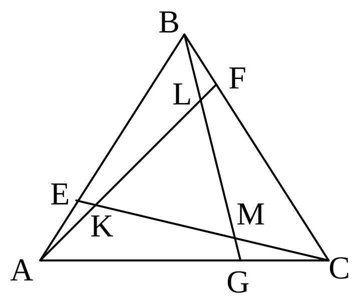


17. ES = 1/2, PS = $\sqrt{3}$ $\Rightarrow$ EP = $\sqrt{13}$/2 and area of ΔEPD = $\sqrt{3}$/2. Now ΔESP is similar to ΔEQS, so ES/EP= EQ/ES $\Rightarrow$ EQ = 1/(2$\sqrt{13}$)$\Rightarrow$ PQ = EP – EQ = 6/$\sqrt{13}$.
Now area of ΔQPR/area of ΔEPD = $(PQ/EP)^2$ = $(12/13)^2$.
So area of ΔQPR = 72$\sqrt{3}$/169 and area of QRDE = 25$\sqrt{3}$/338.

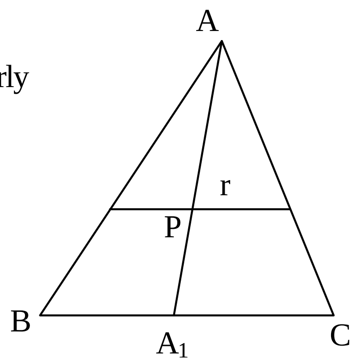


18. Let the angle bisector of $\angle$BAC intersect DE at F and BC at G. By symmetry, AG is the perpendicular bisector to DE and BC. Now AF = $\sqrt{7-\frac{1}{4}}=\frac{3\sqrt{3}}{2}$.
If AB = x, then AG = $\frac{x\sqrt{3}}{2}$. So

$(x-3)\frac{\sqrt{3}}{2}$ = AG – AF = FG = $\sqrt{2^2-\left(\frac{x-1}{2}\right)^2}$ $\Rightarrow$ x = $\frac{5+\sqrt{13}}{2}$. (The case where the positions of D, E are reversed will lead to D, E on edge BC.)



19. Let [XYZ] denote the area of ΔXYZ. Since [ABC] =1, [AEC] = AE/AB = 1/4. Now [ACK]/[ABK] = FC/BF= 3 and [ABK]/[AEK] = 4 $\Rightarrow$ [ACK]/[AEK] = 12$\Rightarrow$ [ACK] = 12[AEC]/13 = 3/13. Similarly [BLK] = [CMB] = 3/13. Then [KLM] = [ABC] – ([ACK] + [BLK] + [CMB]) = 4/13.



20. Let the triangle have vertices A, B, C and P be the point inside. Define A_1 to be the intersections of lines AP and BC and similarly define B_1 and C_1. Suppose the intercepted segments through P have length r. Then $1-\frac{r}{a}=1-\frac{AP}{AA_1}=\frac{A_1P}{AA_1}=\frac{[BPC]}{[ABC]}$
Since [BPC] + [CPA] + [APB] = [ABC], we have $\left(1-\frac{r}{a}\right)+\left(1-\frac{r}{b}\right)+\left(1-\frac{r}{c}\right)=1$

$\Rightarrow$ r = $\frac{2abc}{ab+bc+ca}$.

Contest 98-99 Solutions

1. Try a few cases and proceed by induction, we obtain the answer $1998^2 = 3992004$.

2. Note that $k(k-1)/2$ is even if and only if $k = 4m$ or $k = 4m + 1$. Thus

$$H_{4m-1} + H_{4m} = -\frac{(4m-1)(4m-2)}{2} + \frac{4m(4m-1)}{2} = 4m - 1; \text{ and}$$

$$H_{4m+1} + H_{4m+2} = \frac{(4m+1)(4m)}{2} - \frac{(4m+2)(4m+1)}{2} = -4m - 1.$$

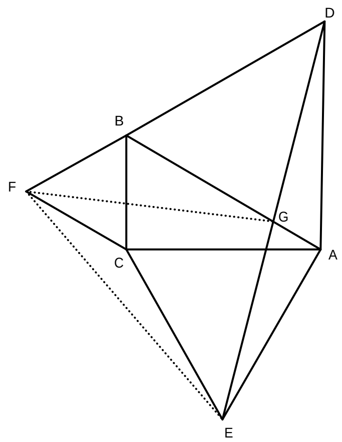
Hence $H_{4m-1} + H_{4m} + H_{4m+1} + H_{4m+2} = -2$.

Therefore, $H_{19} + H_{20} + \ldots + H_{98} = 20(-2) = -40$.

3. There are 15 chapters with odd number of pages and a chapter after this kind must start from a page of different parity, thus parity change at least 14 times. This means that we have at least 7 chapters starting from an even-numbered page. Hence we have at most 23 = 30 – 7 chapters starting from an odd-numbered page.

4. Divide the carpet into 41^2 square of size 3×3 and consider the central square of any of these 3×3 square. No matter whether it is a red or blue square, we have 4 red and 5 blue squares in this 3×3 squares. Hence the total number of red squares $= 4 \times 41^2 = 6724$.

5. Let [XYZ] be the area of ΔXYZ, we have BC = 1/2, AC $= \sqrt{3}/2$, $\angle EAG = 90^o$, $\angle ECF = 150^o$, and D, B, F are collinear.

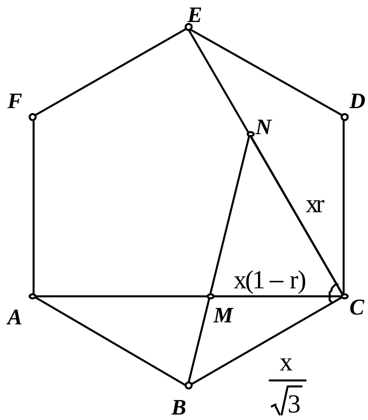


$[AEG] = (\sqrt{3}/4)AG = [ADG]$, so GE = GD.

Then $[DGF] = [EGF] = \frac{1}{2}[DEF] = \frac{1}{2}([ABC] + [ABD] + [BCF] + [ACE] + [CEF] - [ADE])$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} + \frac{3\sqrt{3}}{16} + \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{8}\right) = \frac{9\sqrt{3}}{32}$$

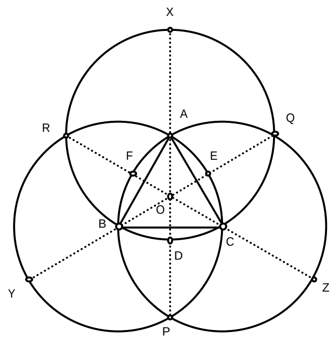
6. Let AC = x, then BC $= \frac{x}{\sqrt{3}}$, CN = xr, CM = x(1– r). Let [XYZ] denote the area of triangle XYZ. Since $\angle NCM = 60^o$, $\angle BCM = 30^o$, and [BCM] + [CMN] = [BCN], we have



$$\frac{x^2(1-r)\sin 30^\circ}{2\sqrt{3}} + \frac{x^2 r(1-r)\sin 60^\circ}{2} = \frac{x^2 r}{2\sqrt{3}}$$

Cancelling x^2 and solving for r, we get $r = \frac{\sqrt{3}}{3}$.

7. There are 10 possibilities for P, as shown in the figure (excluding A, B and C).



8. Let G be the centroid and H be the midpoint of CG. It may be proved that GD = 3, DH = 4 and HG = 5. Hence GDH is a right-angled triangle and its area = 6. Now this triangle is 1/12 of ΔABC, hence area of ΔABC = 72 sq. units

9. If $a \geq 3$, then $\overline{abcd} \times 4$ is a 5-digit number, thus a = 1 or 2. But $\overline{dcba}$ is even, thus a = 2. Now $\overline{2bcd} \times 4 = \overline{dcb2}$, get $d \geq 8$, and d×4 ends at 2, get d = 8. Hence $\overline{2bc8} \times 4 = \overline{8cb2}$, or 8000 + 400b + 40c + 32 = 8000 + 100c + 10b + 2. That is 13b + 1 = 2c, we have then b = 1, c = 7. Hence the number is 2178.

10. Note that $2xy - 5x + y = 55 \Leftrightarrow (2x + 1)(2y - 5) = 105$. Since 105 has 8 positive and 8 negative factors, there are 16 ordered pairs satisfying this equation.

11. $[10^{99}/(10^{33} + 3)] = [(10^{99} + 3^3 - 3^3)/(10^{33} + 3)] = [10^{66} - 3\times10^{33} + 3^2 - 3^3/(10^{33} + 3)] = 10^{66} - 3\times10^{33} + 3^2 - 1$. The last 3 digits, from left to right are 0, 0, and 8.

12. $x_{n+1}^2 - x_n^2 = 2 + \dfrac{1}{x_n^2}$, thus $x_n^2 - x_0^2 = 2n + \sum_{k=0}^{n-1} \dfrac{1}{x_k^2}$, that is $x_{1998}^2 = 4021 + \sum_{k=1}^{1997} \dfrac{1}{x_k^2}$.

Hence $x^2_{1998} > 4021 > 63^2$. Now x_n is increasing and $\dfrac{1}{x_n^2} < \dfrac{1}{2n+25}$. Thus

$$x_{1998}^2 = 4021 + \left(\frac{1}{x_0^2} + \cdots + \frac{1}{x_{24}^2}\right) + \left(\frac{1}{x_{25}^2} + \cdots \frac{1}{x_{124}^2}\right) + \left(\frac{1}{x_{125}^2} + \cdots + \frac{1}{x_{999}^2}\right) +$$

$$\left(\frac{1}{x_{1000}^2} + \cdots + \frac{1}{x_{1997}^2}\right) < 4021 + 25\times\frac{1}{25} + 200\times\frac{1}{75} + 675\times\frac{1}{475} + 998\times\frac{1}{2025} \approx 4026.58$$

$< (63.5)^2$. Hence the required number is 63.

13. The expression equals 5050. So if x is the sum of all positive terms after changing signs, we have $x - (5050 - x) = 1998$ or $x = 3524$. To minimize number of "–" signs in the modified expression, we add the lowest numbers. Note that 1 + 2 + 3 + …+ 83 = 3486, we achieve 3524 by for instance, adding 84 but subtracting 46, thereby changing 17 "+" signs to "–" signs.

14. Let $m = 1000$, the given sequence is $m, x, m - x, 2x - m, 2m - 3x, 5x - 3m, 5m - 8x, \ldots$. An optimal x satisfies as many of the following inequalities as possible before failing: $x < m$, $m/2 < x$, $x < 2m/3$, $3m/5 < x$, $x < 5m/8$, …. Beginning with the 4th inequality, an optimal x must satisfy $600 < x$, $x < 625$, $615 < x$, $x < 620$, $617 < x$, $x < 619$. It follows that $x = 618$.

15. Using cosine rule, $\tan A = \sin A / \cos A = bc \sin A / bc \cos A = 4S/(b^2 + c^2 - a^2)$, where S = area of ΔABC.
Similarly, $\tan B = 4S/(a^2 + c^2 - b^2)$ and $\tan C = 4S/(a^2 + b^2 - c^2)$.
Hence $1/2 = \tan A/ \tan B = (a^2 + c^2 - b^2)/(b^2 + c^2 - a^2)$,
$1/3 = \tan A/\tan C = (a^2 + b^2 - c^2)/(b^2 + c^2 - a^2)$.
Thus $3b^2 - c^2 = 3a^2$, $b^2 - 2c^2 = -2a^2$. Solve for b and c in terms of a, we have $b = 2a\sqrt{2}/\sqrt{5}$, $c = 3a/\sqrt{5}$ and so $b/c = 2\sqrt{2}/3$, i.e., $AC/AB = 2\sqrt{2}/3$.

16. Write $n = 2^s q$ where q is odd.
Now $1997^n - 1 = (1997^{2^s} - 1)(1997^{2^2(q-1)} + 1997^{2^s(q-2)} + \ldots + 1)$. Hence $2^{1998} \mid 1997^n - 1$ implies $2^{1998} \mid 1997^{2^s} - 1$. So for the smallest n, we may take $q = 1$, i.e. $n = 2^s$.
Now $1997^{2^s} - 1 = (1997^{2^{s-1}} + 1)(1997^{2^{s-2}} + 1)\cdots(1997 + 1)(1997 - 1)$, $1997 - 1 = 2^2 \times 249$, $1997 + 1 = 2 \times 999$, and $1997^{2^k} + 1 \equiv 2 \pmod 4$. So $1997^{2^s} - 1$ is divisible by 2^{s+2} but not by 2^{s+3}. Hence the smallest possible n is 2^{1996}.

17. Suppose 1998 equals the sum of n consecutive integers starting from k, then summing up the arithmetic progression, we have
$1998 = k + (k + 1) + \ldots + (k + n - 1) = n(2k + n - 1)/2$ or $n(2k + n - 1) = 3996$.
Note that exactly one of n and $(2k + n - 1)$ is odd. The only odd factors of 3996 are 1, 3, 9, 27, 37, 111, 333, and 999, therefore either n or $(2k + n - 1)$ takes on one of these 8 values. This leaves us 16 possibilities for the ordered pair (n, k).

18. There are $[1998/11] - [1000/11] = 91$ multiples of 11 from 1001 to 1998. Of these 91 numbers, $[1998/121] - [1000/121] = 8$ are multiples of 121, and only $[1998/1331] - [1000/1331] = 1$ is a multiple of 1331. Thus $k = 91 + 8 + 1 = 100$.

19. Since $84 = 3 \times 4 \times 7$, and since a number is divisible by 4 if and only if the two-digit number formed from its last two digits is divisible by 4, the last two digits are 76. To make the number divisible by 3, we need at least 2 more 7's hence our 1st candidate is 7776. However, this number is not a multiple of 7. Next we consider 77676, 76776 and 67776. Only 76776 is divisible by 7.

20. $2mnp = (m + 2)(n + 2)(p + 2)$ implies $2m/(m + 2) = (n + 2)(p + 2)/np > 1$, which shows that $m \geq 3$.
Next rewrite this equation as $(m - 2)(n - 2)(p - 2) = 8(m + n + p)$.
Let $a = m - 2$, $b = n - 2$, $c = p - 2$. Then $a \geq 1$. And so $abc = 8(a + b + c + 6)$ can be rewritten as $\frac{c}{8} = \frac{a+b+6}{ab-8} \leq \frac{ab+1+6}{ab-8} = \frac{ab-8+15}{ab-8} = 1 + \frac{15}{ab-8}$ because $(a - 1)(b - 1) \geq 0$. This shows that c cannot be larger than $8 \times 16 = 128$, and c attains this value if $ab = 9$ and $a + b = ab + 1 = 10$. Thus $c = 128$, when $a = 1, b = 9$, and $m = 3$, $n = 11$, and $p = 130$ are the dimensions of a possible box.

Contest 99-00 Solutions

1. Notice that $A_k < A_{k+1}$ if and only if $\frac{19^k + 99^k}{k!} < \frac{19^{k+1} + 99^{k+1}}{(k+1)!}$; this can be simplified to $19^k(k-18) + 99^k(k-98) < 0$ which is true if k = 1, 2, . . ., 97 and is false for $k \ge 98$. Hence A_k is increasing for k = 1, 2, . . ., 98 and starts to decrease. Thus the sequence is greatest when n = 98.

2. Taking reciprocals, get $\frac{x(y+z)}{x+y+z} = 2$, $\frac{y(z+x)}{x+y+z} = 3$, $\frac{z(x+y)}{x+y+z} = 4$.

   Summing up, get $\frac{xy+yz+zx}{x+y+z} = \frac{9}{2}$, subtract the first three equations from the last one, get $\frac{yz}{x+y+z} = \frac{5}{2}$, $\frac{zx}{x+y+z} = \frac{3}{2}$, $\frac{xy}{x+y+z} = \frac{1}{2}$. Dividing one of the equations from another get $y = \frac{5}{3}x$ and z = 3y = 5x. Put back into the original equation get $x = \frac{23}{10}$.

3. Denote the five colours as C_1, C_2, C_3, C_4 and C_5 respectively. Start with base ABCD and the faces VAB, VBC, they can be coloured in any three different colours (say C_1, C_2, C_3) giving $5 \times 4 \times 3 = 60$ ways. The other two faces VCD and VDA can be coloured in the following seven ways:

VCD	C_2	C_2	C_2	C_4	C_4	C_5	C_5
VDA	C_3	C_4	C_5	C_3	C_5	C_3	C_4

   Hence the total number of ways = $5 \times 4 \times 3 \times 7 = 420$.

4. Rewrite the equation as $(y^2 - 10)(3x^2 + 1) = 517 - 10 = 507 = 3\times13^2$. Since $3x^2 + 1$ is positive, $y^2 - 10$ is a positive divisor of 3×13^2. Thus $y^2 - 10$ = 1, 3, 13, 39, 169, or 507, or y^2 = 11, 13, 23, 49, 179 or 517, but 49 is the only perfect square, thus $y^2 - 10 = 39$ and $3x^2 + 1 = 13$, which implies $3x^2y^2 = 12\times49 = 588$.

5. From page 1 to 9, 9 digits were used, from page 10 to 99, a total of $2\times90 = 180$ digits were used. There were $2001 - 189 = 1812$ digits remaining, enough to number $1812 \div 3 = 604$ pages. Total number of pages = 9 + 90 + 604 = 703.

6. For n to be largest, $n! = (n+1)n(n-1)\ldots(10)(5)$, gives 24 = n + 1 or n = 23.

7. $\frac{15x-7}{5} \le \frac{5+6x}{8} < \frac{15x-7}{5} + 1 \Rightarrow \frac{41}{90} < x \le \frac{9}{10} \Rightarrow \frac{-1}{15} < 3x - \frac{7}{5} \le \frac{13}{10}$.

   Since $3x - \frac{7}{5}$ is an integer, $3x - \frac{7}{5} = 0$ or 1, giving $x = \frac{7}{15}$ or $\frac{4}{5}$.

8. Let the sides of such triangle be $n-1, n, n+1$. Then $(n-1)+n+(n+1) \le 100$ and $(n-1)+n > n+1$, imply $n > 2$ and $n \le 33$, that is $n = 3, 4, \ldots, 33$. When $n = 3$, $2^2 + 3^2 < 4^2$, the triangle is obtuse; when $n = 4$, $3^2 + 4^2 = 5^2$, the triangle is right; when $n \ge 5$, $n^2 + (n-1)^2 - (n+1)^2 = n^2 - 4n = n(n-4) > 0$, the triangle is acute. Thus the required number is $33 - 4 = 29$.

9. Draw CE // BD to meet AD produced at E. Since AC = 5, CF = 4, we have AF = 3. Using similar triangles ($\Delta ACF \sim \Delta AEC$), get $\frac{3}{4} = \frac{AF}{CF} = \frac{AC}{CE} = \frac{5}{CE}$, hence $CE = \frac{20}{3}$. Since area of ΔABC = area of ΔCDE, we have finally area of ABCD = area of $\Delta ACE = 5 \times \frac{20}{3} \times \frac{1}{2} = \frac{50}{3}$.

10. Let EC = x, BE = y, ED = z. Since BC = CD, $\angle BAC = \angle DAC$, thus $\Delta DCE \sim \Delta ACD$. Hence $\frac{CD}{CA} = \frac{EC}{DC}$, or $\frac{4}{6+x} = \frac{x}{4}$, giving $x = 2$.
Now since ABCD is cyclic $AE \times EC = BE \times ED$, we get $6 \times 2 = yz = 12$.
From ΔBCD, $y + z < 4 + 4 = 8$, hence $y = 3, z = 4$ or $y = 4, z = 3$. In each case, $y + z = 7$.

11. Note $a_{n+3} = a_{n+2} - a_{n+1} = (a_{n+1} - a_n) - a_{n+1} = -a_n$, thus $a_{n+6} = -a_{n+3} = -(-a_n) = a_n$, thus the sequence is periodic of period 6. Now $S_n = a_n + a_{n-1} + \ldots + a_1 = (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + \ldots + (a_2 - a_1) + a_2 + a_1 = a_{n-1} + a_2$. Thus $S_{1588} = a_{1587} + a_2 = a_3 + a_2 = 1997$, $S_{1997} = a_{1996} + a_2 = a_4 + a_2 = (a_3 - a_2) + a_2 = 1588$, hence $a_2 = 1997 - 1588 = 409$. Finally, $S_{1999} = S_{1998} + a_2 = a_6 + a_2 = -a_3 + a_2 = -1588 + 409 = -1179$.

12. Clearly $n < 7!$. Let $n = a\times 6! + b\times 5! + c\times 4! + d\times 3! + e\times 2! + f\times 1!$ with $a \le 6$, $b \le 5$, $c \le 4$, $d \le 3$, $e \le 2$, $f \le 1$. Put into the original equation, gets $1237a + 206b + 41c + 10d + 3e + f = 1999$. Solving get $a = 1, b = 3, c = 3, d = 2, e = 0, f = 1$, and $n = 1165$.

13. First, $\frac{31}{7} < \frac{n+k}{n} = 1 + \frac{k}{n} < \frac{15}{8}$ gives $48n < 56k < 49n$ and there are $n - 1$ integers in $(48n, 49n)$. If $n - 1 \ge 2\times 56 = 112$, then there are at least 2 multiples of 56. So $n = 112$ is the largest candidate. Indeed, $48\times 112 = 56\times 96 < 56\times 97 < 56\times 98 = 49\times 112$. Hence $n = 112$ is the answer.

14. We have $\frac{1}{2}n(n+1) < 1922$ and $\frac{1}{2}n(n+1) + n \ge 1922$, giving $n^2 + n - 3844 < 0$ and $n^2 + 3n - 3844 \ge 0$. Using $\sqrt{15377} \approx \sqrt{15376} = 124$, the first inequality implies $-62.5 < n < 61.5$, and the second implies $n \le -63.5$ or $n \ge 60.5$. Combine the two inequalities, get $n = 61$. Since $1 + 2 + 3 + \ldots + 61 = 1891$, the extra number added is $1922 - 1891 = 31$.

15. Let S be the sum of the remaining numbers. Then $\frac{S}{n-4} = 51.5625 = \frac{825}{16}$. Hence 16S = $825(n-4) \Rightarrow \frac{n-4}{16}$ is an integer $\Rightarrow$ n = 16k + 4.

Now $\frac{(1+2+\cdots+n)-(n+(n-2)+(n-4)+(n-6))}{n-4} = \frac{n^2-7n+24}{2(n-4)} \le \frac{825}{16}$

$\Rightarrow 8.2 < n < 106$.

Also, $\frac{(1+2+\cdots+n)-(2+4+6+8)}{n-4} = \frac{n^2+n-40}{2(n-4)} \ge \frac{825}{16} \Rightarrow n < 7.5$ or $n > 98.3$.

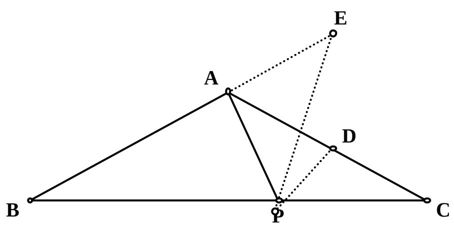
For n = 16k + 4 lies between 98.3 and 106, n = 100.

Thus, $S = \frac{825}{16} \times 96 = 825 \times 6 = 4950$. Note 1 + 2 + …+ 100 = 5050.

If x is the largest number removed, then x + (x – 2) + (x – 4) + (x – 6) = 5050 – 4950 = 100, giving x = 28.

16. In base 3, the sequence is 1, 10, 11, 100, 101, 110, 111, …. Since 200 in base 2 is $(11001000)_2$, hence the 200 th term in base 3 is $(11001000)_3 = 2943$.

17. From the diagram, AE = AP = AD = 8, so that ∠APD = ∠ADP, ∠AEP = ∠APE, and ∠BAP = 2∠AEP. Now ∠APD = ∠APC – ∠CPD = ∠B + ∠BAP – ∠CPD = ∠B + 2∠AEP – ∠CPD, ∠ADP = ∠C + ∠CPD = ∠B + ∠CPD, as ∠APD = ∠ADP, we have ∠AEP = ∠CPD. Thus ΔCPD ~ ΔBEP.



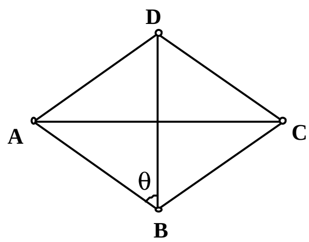
Hence $\frac{CP}{CD} = \frac{BE}{BP}$ or CP×BP = CD×BE = (12 – 8)×(12 + 8) = 80.

18. Let θ = ∠ABD, then AC + BD = 10(sinθ + cosθ) = $10\sqrt{2}\sin(\theta + 45^o)$. Now $45^o < \cos^{-1}\frac{3}{5} \le \theta < 90^o$

$\Rightarrow 90^o < \cos^{-1}\frac{3}{5} + 45^o \le \theta + 45^o < 135^o$.

Hence $\sin(\theta + 45^o)$ is maximum when $\theta = \cos^{-1}\frac{3}{5}$.

Thus AC + BD = $10\left(\frac{3}{5}+\frac{4}{5}\right) = 14$.



19. No unit squares at the “boundary” are totally covered, there are altogether $12\times2 + 10\times2 = 44$ such unit squares. For the unit square whose farthest distance is $\sqrt{5^2+5^2}$ from the origin, it is not covered, since $\sqrt{5^2+5^2} = 5\sqrt{2} > 7 > 6$. There are altogether 4 such squares. For the squares whose “distance” is $\sqrt{4^2+5^2}$ from the origin, it is not covered, since $\sqrt{4^2+5^2} = \sqrt{41} > 6$. There are 8 such squares. All other squares are covered. And hence we get $144 - 44 - 4 - 8 = 88$ squares covered by the circle.

20. Partition the integers into subset $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{4, 5, 6\}$, $A_4 = \{7, 8, 9, 10\}$, $A_5 = \{11, 12, \ldots, 16\}$, $A_6 = \{17, 18, \ldots, 25\}$, $A_7 = \{26, 27, \ldots, 39\}$, $A_8 = \{40, 41, \ldots, 60\}$, $A_9 = \{61, 62, \ldots, 91\}$. Choose one member from each group, then the ratio of any two of them will exceed $\frac{3}{2}$ or less than $\frac{2}{3}$, thus $k > 9$. On the other hand, if 10 numbers are chosen, two must be from the same group and will have a ratio bounded by $\frac{2}{3}$ and $\frac{3}{2}$. Thus the minimum value of k is 10.

Contest 00-01 Solutions

1. Let $a = 2^x - 4$, $b = 4^x - 2$, then the equation becomes
$a^3 + b^3 = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \Leftrightarrow 0 = ab(a + b) \Leftrightarrow a = 0$ or $b = 0$ or $a + b = 0$.
Thus $2^x - 4 = 0$ or $4^x - 2 = 0$ or $4^x + 2^x - 6 = (2^x + 3)(2^x - 2) = 0$, we have $x = 2, \frac{1}{2}$ or 1. The required sum $= \frac{7}{2}$.

2. The numerator and the denominator are relatively prime, thus any prime factor must occur entirely in the numerator or in the denominator (but not both). There are 10 prime factors of 30!, i.e. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. For each of these, one put it up or down, thus 2^{10} choices, exactly half of them will have values between 0 and 1, i.e., $2^9 = 512$ choices.

3. x^{17} can only be obtained by multiplying two x^5' s and one x^7. There are 20 ways to get x^7 and $C_2^{19} = 171$ ways to get two x^5' s in the remaining 19 factors. So the answer is $20\times171 = 3420$.

4. Note $\left[\frac{2000\times1999}{2001}\right] = \left[1999 - \frac{1999}{2001}\right] = 1999 - 1 - \left[\frac{1999}{2001}\right]$.
Hence $\left[\frac{1\times1999}{2001}\right] + \left[\frac{2000\times1999}{2001}\right] = 1999 - 1 = 1998$. Similarly, we have
$\left[\frac{2\times1999}{2001}\right] + \left[\frac{1999\times1999}{2001}\right] = 1998, \ldots, \left[\frac{1000\times1999}{2001}\right] + \left[\frac{1001\times1999}{2001}\right] = 1998$.
Summing up, we get the sum = 1998000.

5. From $x^2 + xy + y^2 = 0$, get $\frac{xy}{(x+y)(x+y)} = 1$. Let $t = \frac{x}{x+y}$, then $\frac{1}{t} = \frac{y}{x+y}$ and $t + \frac{1}{t} = 1$, $t^2 - t + 1 = 0 = (t + 1)(t^2 - t + 1) = t^3 + 1 \Rightarrow t^3 = -1$. Thus $t^{2001} = t^{3(667)} = -1$, and hence $t^{2001} + \frac{1}{t^{2001}} = -2$.

6. Let m and n be integral roots of $x^2 + ax + 8a = 0$, with $m \le n$.
Then $x^2 + ax + 8a = (x - m)(x - n) = x^2 - (m + n)x + mn$. We have $a = -(m + n)$, and $8a = mn$. Thus a is an integer and $-8(m + n) = mn$, get $mn + 8m + 8n = 0$ or $(m + 8)(n + 8) = 64$. Consider the factorizations of $64 = 64\times1 = 32\times2 = 16\times4 = 8\times8 = 4\times16 = 2\times32 = 1\times64 = (-64)\times(-1) = (-32)\times(-2) = (-16)\times(-4) = (-8)\times(-8) = (-4)\times(-16) = (-2)\times(-32) = (-1)\times(-64)$, we obtain 8 distinct pairs of m and n.

7. $\tan\alpha + \tan\beta = -\pi$, $\tan\alpha\tan\beta = \sqrt{2} \Rightarrow \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\pi}{\sqrt{2} - 1}$.

Now $\quad \sin^2(\alpha + \beta) + \pi\sin(\alpha + \beta)\cos(\alpha + \beta) + \sqrt{2}\cos^2(\alpha + \beta)$
$= \cos^2(\alpha + \beta)\,[\tan^2(\alpha + \beta) + \pi\tan(\alpha + \beta) + \sqrt{2}$

$$= \frac{1}{1+\tan^2(\alpha+\beta)}[\tan^2(\alpha+\beta)+\pi\tan(\alpha+\beta)+\sqrt{2}]$$

$$= \frac{(\sqrt{2}-1)^2}{\pi^2+(\sqrt{2}-1)^2}\left[\frac{\pi^2}{(\sqrt{2}-1)^2}+\frac{\pi^2}{\sqrt{2}-1}+\sqrt{2}\right]=\sqrt{2}$$

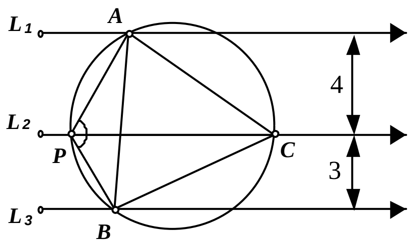
8. After the 2000th operation, only those lamps with number which have an odd number of factors will be on. This is equal to the number of perfect squares less than 2000. Since $44^2 = 1936$, and $45^2 = 2025$. Therefore the number of lamps which are on = 44.

9. Let m be the number of sides of the polygon determined by A_n, A_1 and B. The degree measures of the interior angles of the three polygons are $180-\frac{360}{n}$, 60 and $180-\frac{360}{m}$ respectively. If 6 < n, the polygon fit together at their common vertex A_1, then

$$360 = 180-\frac{360}{n}+60+180-\frac{360}{m} \Rightarrow n=\frac{6m}{m-6}=6+\frac{36}{m-6}.$$

So m > 6, and n is a decreasing function of m. The largest value of n is 42, when m = 7.

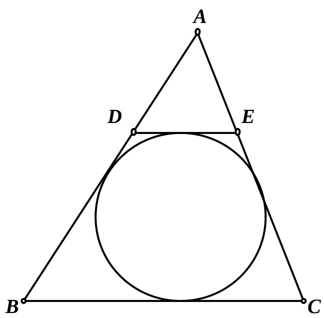
10. The triangle is constructed by drawing $\angle APC = \angle BPC = 60^o$ on the lines, and constructing APBC as circumcircle of ΔAPB. Then $\angle ABC = \angle APC = 60^o$ and $\angle BAC = \angle BPC = 60^o$. Now $\frac{4}{AP} = \sin 60^o = \frac{\sqrt{3}}{2} \Rightarrow AP = \frac{8}{\sqrt{3}}$, and $\frac{3}{BP} = \sin 60^o = \frac{\sqrt{3}}{2} \Rightarrow AP = \frac{6}{\sqrt{3}}$. $(AB)^2 = \frac{64}{3}+\frac{36}{3}-2\cdot\frac{8}{\sqrt{3}}\cdot\frac{6}{\sqrt{3}}\cos 120^\circ = \frac{148}{3}$. Hence area of $\Delta ABC = \frac{\sqrt{3}}{4}\cdot\frac{148}{3} = \frac{37\sqrt{3}}{3}$.



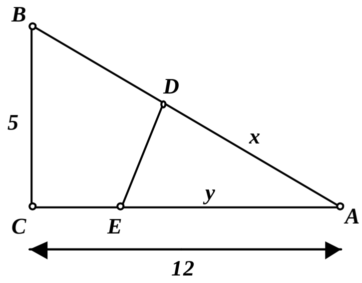
11. Let DE = x, BC = a, then perimeter of $\Delta ADE = p - 2a$ (tangent property). As $\Delta ADE \sim \Delta ABC$, then $\frac{x}{a}=\frac{p-2a}{p}$,

get $x = \frac{a(p-2a)}{p} = \frac{2}{p}\left(\frac{1}{2}ap-b^2\right) = \frac{2}{p}\left[\left(\frac{p}{4}\right)^2-\left(a-\frac{p}{4}\right)^2\right]$.

When $a=\frac{p}{4}$, x attains its maximum. Thus maximum value of $DE = \frac{2}{p}\left(\frac{p}{4}\right)^2 = \frac{p}{8}$.



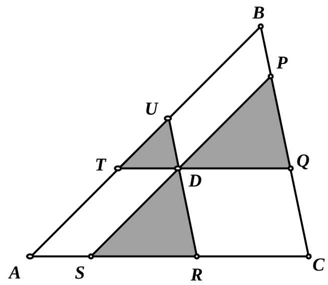
12. Area of $\Delta ABC = \frac{1}{2}\times 5\times 12 = 30$, $\sin A = \frac{5}{13}$. Let AD = x, AE = y, area of $\Delta ADE = \frac{1}{2}xy\sin A = 15$. Hence xy = 78.



By cosine formula, $DE^2 = x^2 + y^2 - 2xy\cos A =$

$(x-y)^2 + 2xy(1-\cos A) = (x-y)^2 + 2\times 78\left(1-\frac{12}{13}\right) = (x-y)^2 + 12 \geq 12$. Thus minimum value of $DE = \sqrt{12}$.

13. Let area of ΔABC = S. Note that $\Delta TUD \sim \Delta ABC$ and their areas are in the ratio $UD^2 : BC^2$. Therefore $\frac{\sqrt{8}}{\sqrt{S}} = \frac{UD}{BC}$, Similarly, $\frac{\sqrt{128}}{\sqrt{S}} = \frac{PQ}{BC}$ and $\frac{\sqrt{32}}{\sqrt{S}} = \frac{DR}{BC}$. Furthermore, BC = BP + PQ + QC = UD + PQ + DR. Therefore,



$$\frac{\sqrt{8}+\sqrt{128}+\sqrt{32}}{\sqrt{S}} = \frac{UD+PQ+DR}{BC} = 1.$$

Hence $\sqrt{S} = 14\sqrt{2}$ or S = 392.

14. Observe that $|y_k - y_{k+1}| = \left|\frac{x_1 + x_2 + \cdots + x_k}{k} - \frac{x_1 + \cdots + x_{k+1}}{k+1}\right| = \left|\frac{x_1 + \cdots + x_k - kx_{k+1}}{k(k+1)}\right|$

$$= \left|\frac{x_1 - x_2 + 2x_2 - 2x_3 + 3x_3 - \cdots + kx_k - kx_{k+1}}{k(k+1)}\right| \leq \frac{|x_1 - x_2| + 2|x_2 - x_3| + \cdots + k|x_k - x_{k+1}|}{k(k+1)}$$

Hence $|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|$

$$\leq \sum_{k=1}^{1999}\frac{|x_1 - x_2| + 2|x_2 - x_3| + \cdots + k|x_k - x_{k+1}|}{k(k+1)} = \sum_{k=1}^{1999}\sum_{i=1}^{k}\frac{i|x_i - x_{i+1}|}{k(k+1)} = \sum_{i=1}^{1999}\sum_{k=i}^{1999}\frac{i|x_i - x_{i+1}|}{k(k+1)}$$

$$= \sum_{i=1}^{1999} i|x_i - x_{i+1}|\left(\sum_{k=i}^{1999}\frac{1}{k(k+1)}\right) = \sum_{i=1}^{1999} i|x_i - x_{i+1}|\left(\frac{2000-i}{2000i}\right) = \sum_{i=1}^{1999}|x_i - x_{i+1}|\left(1-\frac{i}{2000}\right)$$

$$\leq \left(1-\frac{1}{2000}\right)\sum_{i=1}^{1999}|x_i - x_{i+1}| = 1999.$$

Take $x_1 = 2000$, $x_2 = x_3 = \ldots = x_{2000} = 0$, we observe that the estimate can be achieved.

15. Consider drawing the circles in the order C_8, C_7, …, C_1 and then locating the points in the circles. Draw C_8 and 8 points will be marked on the circle. Draw C_7 so that it pass through 2 of the 8 points already exist, and 5 new points will have to be created. Now draw C_6 so that it passes through 2 existing points on C_8 and 2 existing points on C_7, and this leaves 2 more points to be created. Now the situation is: we have drawn C_8, C_7, C_6 with 6 points fixed and 9 points (4 on C_8 only, 3 on C_7 only, 2 on C_6 only) that can be fixed at later stage. Now attempt to draw C_5 and C_4 by suitably selecting the positions of these 9 points. It may be observed that there will be no difficulty to draw C_3 and C_2 by selecting three or two existing points not on the same circle. Finally, C_1 may be drawn to passes through any one existing point. The minimum number of points is 8 + 5 + 2 = 15.

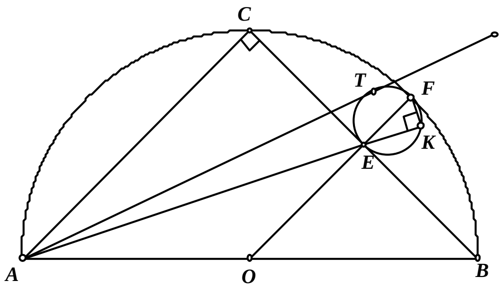
16. Denote the number of ‘b,b’ pairs in $f^{(n)}(a)$ by P_n and the number of ‘a, b’ pairs by Q_n. The number of ‘b,b’ pairs in $f^{(n)}(a)$ is equal to the number of ‘a, b’ pairs in $f^{(n-1)}(a)$. The number of ‘a, b’ pairs in $f^{(n-1)}(a)$ is equals to the number of ‘a’s plus the number of ‘b, b’ pairs in $f^{(n-2)}(a)$. Moreover $f^{(n-2)}(a)$ consists of 2^{n-2} letters, and half of them are ‘a’s. Therefore, $P_n = Q_{n-1} = 2^{n-3} + P_{n-2}$.

Inductively, we have $P_n = 2^{n-3} + 2^{n-5} + \cdots + \begin{cases} 2^0 + P_1 & \text{n odd} \\ 2^1 + P_2 & \text{n even} \end{cases}$.

Now $P_1 = 0$, $P_2 = 1$. It follows that if n is odd, $P_n = 2^{n-2} + 2^{n-5} + \ldots + 1 = \dfrac{2^{n-1}-1}{3}$, if n is even $P_n = 2^{n-3} + 2^{n-5} + \ldots + 2 + 1 = \dfrac{2(2^{n-3}-1)}{3} + 1 = \dfrac{2^{n-1}+1}{3}$.

Combining the results, we obtain $P_n = \dfrac{2^{n-1}+(-1)^n}{3}$.

17. Let T be on Γ so that AT is tangent to Γ as shown. Now O, E, F are collinear, EF is a diameter of Γ. Let AE meets Γ at K. Then ΔEKF ~ ΔACE ($\angle C = \angle K = 90^o$, $OE \perp BC \Rightarrow OE // AC \Rightarrow \angle CAE = \angle KEF$). So $\dfrac{EK}{AC} = \dfrac{EF}{AE}$

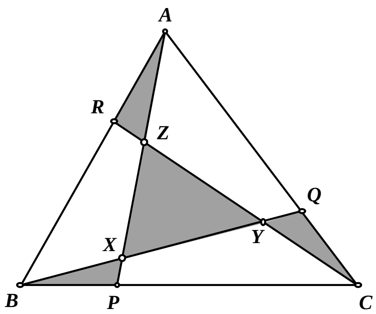


$$\Rightarrow EK = \frac{AC \cdot EF}{AE} = \frac{AC(OF-OE)}{\sqrt{AC^2+CE^2}}$$

$$= \frac{(2\sqrt{2})(2-\sqrt{2})}{\sqrt{8+2}} = \frac{4-2\sqrt{2}}{\sqrt{5}}.$$

$$AK = AE + EK = \frac{4+3\sqrt{2}}{\sqrt{5}} \Rightarrow AT = \sqrt{AE \cdot AK} = \sqrt{6+4\sqrt{2}} = 2+\sqrt{2}.$$

18. Let [XYZ] denote the area of ΔXYZ. Let AR : RB = BP : PC = CQ : QA = 1 : α.



$$\frac{[ACR]}{[ABC]} = \frac{1}{1+\alpha}; \ \frac{[ABZ]}{[ACZ]} = \frac{1}{\alpha}; \ \frac{[ARZ]}{[ABZ]} = \frac{1}{1+\alpha}$$

$$\therefore \frac{[ARZ]}{[ACZ]} = \frac{1}{\alpha(\alpha+1)} \text{ and } \frac{[ARZ]}{[ACR]} = \frac{1}{\alpha^2+\alpha+1}.$$

Thus $\dfrac{[ARZ]}{[ABC]} = \dfrac{1}{(\alpha+1)(\alpha^2+\alpha+1)}$ ----- (1);

$$\frac{[ACZ]}{[ACR]} = \frac{\alpha^2+\alpha}{\alpha^2+\alpha+1} \text{ and } \frac{[ACZ]}{[ABC]} = \frac{\alpha}{\alpha^2+\alpha+1}.$$

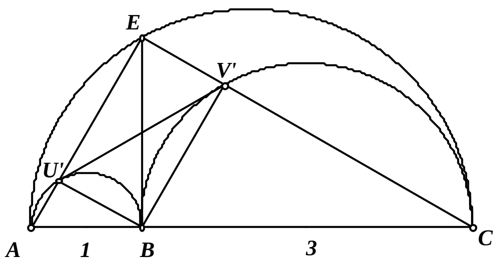
$\therefore \dfrac{[XYZ]}{[ABC]} = 1 - \dfrac{3\alpha}{\alpha^2+\alpha+1} = \dfrac{\alpha^2-2\alpha+1}{\alpha^2+\alpha+1}$ ------ (2). Combining (1) and (2), we have

$\dfrac{\alpha^2-2\alpha+1}{\alpha^2+\alpha+1} = \dfrac{1}{(\alpha+1)(\alpha^2+\alpha+1)} \Rightarrow (\alpha+1)(\alpha^2-2\alpha+1) = 1 \Rightarrow \alpha^2 - \alpha - 1 = 0$

$\Rightarrow \alpha = \dfrac{1+\sqrt{5}}{2}$. As [ARZ] = 1 cm^2, we have $[ABC] = (\alpha+1)(\alpha^2+\alpha+1)$, and hence

$$[PCYX] = \frac{1}{1+\alpha}(\alpha + 1)(\alpha^2 + \alpha + 1) - 2 = \alpha^2 + \alpha - 1 = 2\alpha = (1 + \sqrt{5})\ \text{cm}^2.$$

19. Let AE intersects Γ_2 at U′, CE intersects Γ_3 at V′. Since $\angle AEC = \angle AU'B = \angle BV'C = 90^o$, EU′BV′ is a rectangle. $\angle V'U'B = \angle EBU' = \angle EAB$ which implies V′U′ is tangent to Γ_2 at U′. Similarly, U′V′ is tangent to Γ_3 at V′, so U = U′, V = V′, and ΔEUV ~ ΔECA. Then



$$\frac{\text{area of } \Delta EUV}{\text{area of } \Delta EAC} = \frac{UV^2}{AC^2} = \frac{EB^2}{AC^2} = \frac{AB \times BC}{AC^2} = \frac{3}{16}.$$

20. For the persons in the photographs we draw family tree. The 0^{th} row corresponds to persons where mothers do not appear in the photographs. Denote by r_k the numbers of persons in the middle of the pictures that are in the k^{th} row, t_k numbers of other persons in the k^{th} row. The totality of mothers of women in row k is s_k. We have $s_{k+1} \le \frac{1}{2} r_{k+1} + t_{k+1}$ (every middle women has a sister in the pictures), and $r_k \le s_{k+1}$ (every women has a daughter in the pictures). Thus $r_k \le \frac{1}{2} r_{k+1} + t_{k+1}$, k = 0, 1, 2, …. Also $1 \le \frac{1}{2} r_0 + t_0$.

Sum up k from 0 to 1, 2, …, we get $(r_0 + r_1 + \ldots) + 1 \le \frac{1}{2}(r_0 + r_1 + \ldots) + (t_0 + t_1 + \ldots)$.

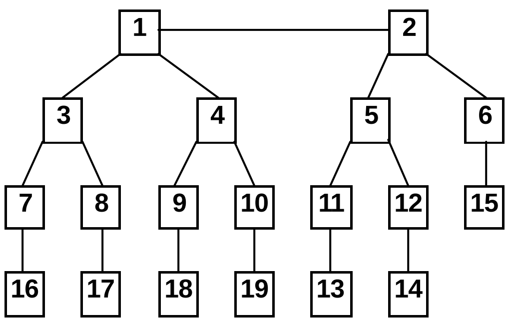
Hence $(r_0 + r_1 + \ldots) + (t_0 + t_1 + \ldots) \ge \frac{3}{2}(r_0 + r_1 + \ldots) + 1 = \frac{3}{2} \times 12 + 1 = 19$. We have such an example:

$r_0 = 2$, $t_0 = 0$

$r_1 = 4$, $t_1 = 0$, $s_1 = 2$

$r_2 = 6$, $t_2 = 1$, $s_2 = 4$

$r_3 = 0$, $t_3 = 6$, $s_3 = 6$



1, 2, 3, …, 12 are middle women, 1 and 2 are sisters. The pictures are (3, 1, 2), (5, 2, 1), (7, 3, 4), (9, 4, 3), (11, 5, 6), (15, 6, 5), (16, 7, 8), (17, 8, 7), (18, 9, 10), (19, 10, 9), (13, 11, 12), (14, 12, 11).

Contest 01-02 Solutions

1. For a number to be divisible by 11, the rule is the sum of the digits in the odd positions minus the sum of the digits in the even positions should be a multiple of 11. So we need (2n + 7 + 9) – (n + 2) = n + 14 divisible by 11. Hence the least n is 8. (Briefly, the rule is because $10^k = (11-1)^k = 11m + (-1)^k$ by the binomial theorem and so $N = a_m 10^m + \ldots + a_1 10 + a_0 = 11M + a_m(-1)^m + \ldots + a_1(-1) + a_0 = 11M + (a_0 + a_2 + \ldots) - (a_1 + a_3 + \ldots)$)

2. Suppose 1 armchair cost \$ a, 1 bookcase cost \$ b and 1 cabinet cost \$ c. Then 8a + 11b + 2c = 875 and 3a + 2b + 5c = 343. Taking the first equation and adding it to 3 times the second equation, we get 17a + 17b + 17c = 1904. So a + b + c = 112.

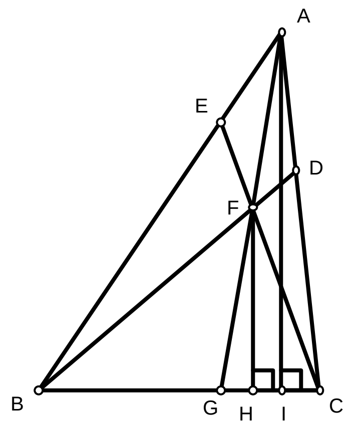
3. $T_1 + T_2 + T_3 + \ldots + T_{2002}$
   $= F(1^2) - F(1) + \ldots + F(2002^2) - F(2002) = F(1^2) + \ldots + F(2002^2) - F(1) - \ldots - F(2002)$
   $= 200(1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0) + 1 + 4 - 200(1 + 2 + \ldots + 9 + 0) - 1 - 2$
   $= 2$

4. $2^{2002}(2^{2003} - 1) = 4\times 16^{500}(8\times 16^{500} - 1)$. Now $M = 16^{500} - 1$ is divisible by 16 – 1 = 15. Thus $2^{2002}(2^{2003} - 1) = 4(M + 1)(8M + 7) = 32M^2 + 60M + 28$ has last two digits equal 28 because $32M^2$ and 60M are divisible by 4×5×5 = 100.

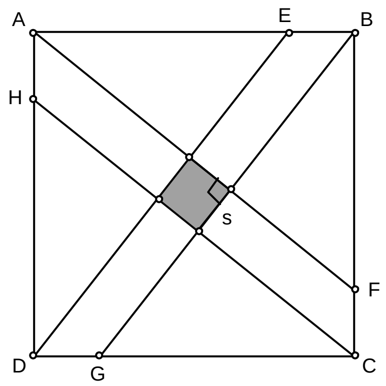
5. The equations can be put in the form
   $$z^2 = x^2 + y^2 - 2xy\cos 30^\circ \text{ and } y^2 = x^2 + z^2 - 2xz\cos 45^\circ,$$
   which remind us the cosine law. So we consider ΔABC with AB = z, BC = x, CA = y, $\angle B = 45^\circ$ and $\angle C = 30^\circ$. By sine law, $y : z = \sin\angle B : \sin\angle C = \sqrt{2} : 1$.

6. The equation can be rewritten as $(x^3 - b)^2 = 100$, which implies $x = \sqrt[3]{b \pm 10}$. Thus $2 = \sqrt[3]{b+10} - \sqrt[3]{b-10}$. Cubing both sides, we get
   $$8 = b + 10 - 3\sqrt[3]{(b+10)^2(b-10)} + 3\sqrt[3]{(b+10)(b-10)^2} - (b - 10)$$
   $$8 = 20 - 3\sqrt[3]{b^2-100}\left(\sqrt[3]{b+10} - \sqrt[3]{b-10}\right) = 20 - 6\sqrt[3]{b^2-100}.$$
   Then $\sqrt[3]{b^2-100} = 2$, which implies $b = \sqrt{108} = 6\sqrt{3}$.

7. Upon drawing an accurate figure and using a protractor, $\angle GFC$ seems to be 20^o. To confirm this, let H, I be the feet of perpendiculars from F and A to BC respectively. Now $\angle BCA = 180^o - \angle BAC - \angle ABC = 80^o$ and $BI = BA \cos 60^o = \frac{1}{2} BA$. Applying sine law to ΔABC, we get $\frac{BA}{\sin 80^o} = \frac{BC}{\sin 40^o}$. So $BA = \frac{BC \sin 80^o}{\sin 40^o} = 2\ BC \cos 40^o$ and $BI = BC \cos 40^o$. In ΔBFC, $\angle BFC = 70^o = \angle BCF$. So BC = BF. Then $BI = BF \cos 40^o = BH$ yielding H = I. Then $AF \perp BC$. So G = H = I and $\angle GFC = \angle HFC = 20^o$.



8. Since the figure is the same after 90^o, 180^o or 270^o rotation about the centre of ABCD, the region must be a square. Let this square has side length s. Now $AF = \sqrt{AB^2 + BF^2} = \frac{29}{21}$. The area of AFCH is $AH \times AB = \frac{1}{21}$ and is also $AF \times s = \frac{29}{21} s$. So $s = \frac{1}{29}$ and $s^2 = \frac{1}{841}$.



9. Let $w = \frac{1}{2} x$, then $w^2 + y^2 = 1$. So $w = \cos\theta$ and $y = \sin\theta$ for some θ. Then $x^2 + 2xy + 4y^2 + x + 2y = 4 + 4\cos\theta\sin\theta + 2\cos\theta + 2\sin\theta$. Now let $u = \cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45^o)$, then $|u| \le \sqrt{2}$, $u^2 - 1 = 2\cos\theta\sin\theta$. Thus

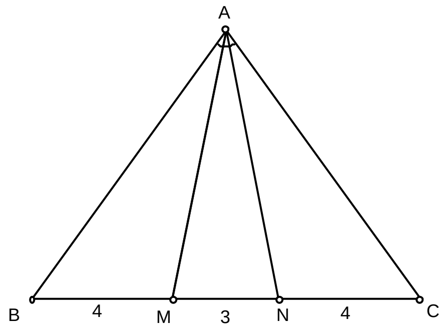
$x^2 + 2xy + 4y^2 + x + 2y = 4 + 2(u^2 - 1) + 2u = 2(u^2 + u + 1) \le 2(2 + \sqrt{2} + 1) = 6 + 2\sqrt{2}$

with equality if $u = \sqrt{2}$, $\theta = 45^o$, $x = \sqrt{2}$, and $y = \frac{\sqrt{2}}{2}$. So the maximum is $6 + 2\sqrt{2}$.

10. Let the length, width and height of the cuboid be L, W, H respectively. If L,W,H are at least two, then $x = (L-2)(W-2)(H-2)$, $y = 2[(L-2)(W-2) + (W-2)(H-2) + (H-2)(L-2)]$ and $z = 4[(L-2) + (W-2) + (H-2)]$. If we let $a = L-2$, $b = W-2$, $c = H-2$, then $1994 = x - y + z - 8 = abc - 2(ab + bc + ca) + 4(a + b + c) - 8 = (a-2)(b-2)(c-2) = (L-4)(W-4)(H-4)$. Since no face is a square, taking all possible factorizations of 1994, the numbers L,W,H are either 1001, 6, 5 or 1001, 3, 2. So LWH = 30030 or LWH=6006.

If one of L,W,H is 1, say H=1, then x=y=0 and $z = 2002 = (L-2)(W-2)$. Now $2002 = 2 \times 7 \times 11 \times 13$. Then 2002 can be factored in 8 ways as product of two positive integers. These give 8 more possible answers for the volume, which are 6012, 4012, 2592, 2392, 2340, 2320, 2232 and 2212.

11. Attempting to draw a figure, it leads to the suspicion that AB = AC. This is the case because the circumradius of ΔBMA being $\frac{BM}{2\sin\angle BAM}$ would equal the circumradius of ΔCAN so that both circumcentres are symmetric with respect to the perpendicular bisector of BC, which must then contain A. Since AM, AN bisect ∠BAN, ∠CAM respectively, we get $\frac{AN}{AB} = \frac{3}{4} = \frac{AM}{AC}$. If AC = x, then $AM = \frac{3}{4}x = AN$. By cosine law, $9 = 2\left(\frac{3}{4}x\right)^2 - 2\left(\frac{3}{4}x\right)^2 \cos\angle MAN$ and $16 = \left(\frac{3}{4}x\right)^2 + x^2 - 2\left(\frac{3}{4}x\right)x\cos\angle NAC$. Since ∠MAN = ∠NAC, solving the equations, we get AC = x = 8.

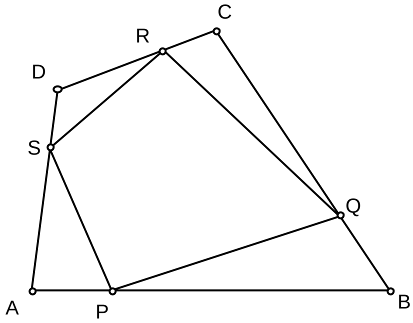


12. Let a_n be the number of n-digit positive integers with the two properties. First, notice that $a_1 = 0$ and $a_2 = 1$. For n ≥ 3, of the a_n integers, there are a_{n-1} of them begin with 2, there are a_{n-1} of them begin with 12 and there are 2^{n-2} of them begin with 11.
So $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for n ≥ 3. Then $a_3 = 3$, $a_4 = 8$, $a_5 = 19$, $a_6 = 43$, $a_7 = 94$, $a_8 = 201$, $a_9 = 423$, $a_{10} = 880$.

13. Let $y = 2001 + n^2$. Without loss of generality, assume n ≥ 0. Since x > y, we have $x = 2001 + (n + 1)^2$. Now, d divides x, y implies d divides x − y = 2n + 1. Then d divides 2y − n(2n + 1) = 4002 − n and 2(4002 − n) + (2n + 1) = 8005. Hence the maximum d can be is 8005. Then n = 4002 + 8005k and $x = 2001 + (4003 + 8005k)^2$. Since $x < 5\times10^7$, k = 0 and $x = 2001 + 4003^2 = 16026010$.

14. Let {x} = x − [x]. Note 0 ≤ {x} < 1. If x > 1, then the equation becomes $x = \frac{\{x\}}{x-1}$. This leads to {x} = x(x − 1) > x − 1 ≥ x − [x] = {x}, a contradiction. Also, if x < −1, then the equation becomes $-x - 2 = \frac{\{x\}}{1-x}$. If x ≤ −3, then {x} = (−x − 2)(1 − x) ≥ 1×4 > {x}, a contradiction. If −3 < x < −2 , then {x} = x + 3 and the equation becomes $-x - 2 = \frac{x+3}{1-x}$, which has the solution $x = -\sqrt{5}$. So the largest possible | x | is $\sqrt{5}$.

15. Let [XY …Z] denote the area of polygon XY …Z. Since $\frac{[SAP]}{[SAB]} = \frac{AP}{AB} = \frac{k}{k+1}$ and $\frac{[SAB]}{[DAB]} = \frac{SA}{DA} = \frac{1}{k+1}$, we get $\frac{[SAP]}{[DAB]} = \frac{k}{(k+1)^2}$.



Similarly, $\frac{[PBQ]}{[ABC]} = \frac{[QCR]}{[BCD]} = \frac{[RDS]}{[CDA]} = \frac{k}{(k+1)^2}$. Now

[PQRS] = [ABCD] – ([SAP] + [PBQ] + [QCR] + [RDS])

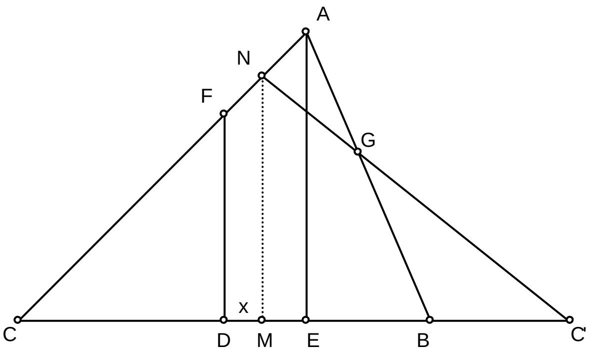
$$= [ABCD] - \frac{k}{(k+1)^2}(\underbrace{[DAB]+[ABC]+[BCD]+[CDA]}_{=2[ABCD]}) = \frac{k^2+1}{(k+1)^2}[ABCD].$$

So $\frac{k^2+1}{(k+1)^2} = 0.52$. Solving for k and noting k < 1, we get $k = \frac{2}{3}$.

16. There are C_3^{30} ways of choosing integers a, x, d such that $1 \le a < x < d \le 30$. The equation a + d = x + x is possible if and only if a and d are both even or both odd, which account for $C_2^{15} + C_2^{15}$ cases. In the remaining $C_3^{30} - 2C_2^{15} = 3850$ cases, we have a + d = x + x′ with x ≠ x′ and a < x, x′ < d. So the answer is $\frac{3850}{2} = 1925$.

17. Suppose $\frac{1}{p} = 0.a_1a_2\cdots a_n\dot{a}_{n+1}\cdots\dot{a}_{n+7}$. Let x and y be the integers with digits $a_1 \ldots a_n$ and $a_{n+1} \ldots a_{n+7}$ respectively. Then $\frac{10^n}{p} = x + \frac{y}{10^7-1}$. So $10^n(10^7 - 1) = p[x(10^7 - 1) + y]$. Then p divides 10 or $10^7 - 1 = 3^2 \times 239 \times 4649$. Since 2, 3, 5 are not such prime, so if p ≠ 4649, then p can only be 239. As $\frac{1}{239} = 0.\dot{0}04184\dot{1}$, the only other choice of p is 239.

18. Consider ΔABC with $a = BC = \frac{4}{3}$, $b = AC = \sqrt{2}$, $c = AB = \frac{\sqrt{10}}{3}$. Let D be the midpoint of BC, E on BC such that AE $\perp$ BC and F on AC such that FD $\perp$ BC. Consider folding along MN $\perp$ BC with M on BC and N on AB or AC. If N is on AB, then the maximum overlapped area occurs when N = A. If N is on CF, then the maximum overlapped area occurs when N = F. So for the problem, we may assume N is on FA. By cosine law, $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{1}{\sqrt{2}}$ and $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{1}{\sqrt{10}}$. Then $\angle C = 45^o$ and $\sin B = \frac{3}{\sqrt{10}}$.



Let C′ be on line BC such that M is the midpoint of CC′ and G be the intersection of AB and NC′. In $\Delta BC'G$, $\angle BGC' = \angle ABC - \angle NC'C = \angle ABC - 45^o$. So $\sin \angle BGC' = \sin B \cos C - \sin C \cos B = \frac{1}{\sqrt{5}}$.

Let x = DM, then $BC' = 2x$ and $C'G = \frac{BC' \sin \angle GBC'}{\sin \angle BGC'} = \frac{2x\left(\frac{3}{\sqrt{10}}\right)}{\frac{1}{\sqrt{5}}} = 3\sqrt{2}x$.

As usual, let [XY …Z] denote the area of polygon XY …Z. Then $[CMN] = \frac{1}{2}\left(\frac{2}{3}+x\right)^2$, $[BC'G] = \frac{1}{2}BC' \times C'G \sin 45^o = 3x^2$ and the overlapped area $[MBGN] = \frac{1}{2}\left(\frac{2}{3}+x\right)^2 - 3x^2$ $= -\frac{5}{2}\left(x-\frac{2}{15}\right)^2 + \frac{4}{15}$, which is maximum when $DM = x = \frac{2}{15}$. (Since $DE = CE - DE = 1 - \frac{2}{3} = \frac{1}{3}$, M is on segment DE and N is on segment FA.) In that case $[MBGN] = \frac{4}{15}$.

International Mathematical Olympiad (IMO)

Hong Kong Teams

29th IMO, July 1988, Canberra, Australia

Chan Shun	Queen's College
Choi Shu-Hung	King's College
Leung Wing-Hong	Queen Elizabeth School
Tai Wai-Ling	Diocesan Boys' School
Yau Shuk-Han	St. Paul's Co-educational College
Yip Nung-Kwan	Tsuen Wan Government Secondary School

30th IMO, July 1989, Braunschweig, West Germany

Chan Kai-Pak	St. Paul's Co-educational College
Chiu Shin-Yeung	King's College
Choi Shu-Hung	King's College
Man Lai-Chee	St. Paul's Co-educational College
Tam Ting-Kin	Kwun Tong Government Secondary School
Yau Shuk-Han	St. Paul's Co-educational College

31st IMO, July 1990, Beijing, China

Chan Kin-Wah	TWGHs Mrs. Wu York Yu Memorial College
Cheng Wing-Leung	Ha Kwai Chung Government Secondary Technical School
Ho Yuk	King's College
Lau Chi-Hin	Pentecostal School
Leung Wing-Kai	Queen's College
Ma Jim-Lok	King's College

32nd IMO, July 1991, Sigtuna, Sweden

Chan Kin-Wah	TWGHs Mrs. Wu York Yu Memorial College
Cheung Cheuk-Wah, Trevor	Queen's College
Ko Chi-Kin. Thomas	Queen's College
Ngai Chi-Ho	Diocesan Boys' School
To Kar-Keung	King's College
Wong Wilkie	Diocesan Boys' School

33rd IMO, July 1992, Moscow, Russia

Chung Wai-Yin	King's College
Lam Chi-Wai	Chuen Yuen College
Lam Pei-Fung	Queen's College
Lee Wai-Fun	St. Mark's School
Suen Yun-Leung	King's College
To Kar-Keung	King's College

34th IMO, July 1993, Istanbul, Turkey

Chan Tsz-Ho	Ying Wa College
Chu Hoi-Pun	Chong Gene Hang College
Lam Chi-Wai	Chuen Yuen College
Lin Kwong-Shing	Tsuen Wan Government Secondary School
Tsui Ka-Hing	Queen Elizabeth School
Yung Fai	Chuen Yuen College

35th IMO, July 1994, Hong Kong

Chu Hoi-Pun	Chong Gene Hang College
Ho Wing-Yip	Clementi Secondary School
Poon Wai-Hoi, Bobby	St. Paul's College
Suen Yun-Leung	King's College
Tsui Ka-Hing	Queen Elizabeth School
Wong Him-Ting	Salesian English School

36th IMO, July 1995, Toronto, Canada

Cheung Kwok-Koon	SKH Bishop Mok Sau Tseng Secondary School
Ho Wing-Yip	Clementi Secondary School
Mok Tze-Tao, Edmond	Queen's College
Poon Wai-Hoi, Bobby	St. Paul's College
Wong Him-Ting	Salesian English School
Yu Chun-Ling	Ying Wa College

37th IMO, July 1996, Mumbai, India

Ho Wing-Yip	Clementi Secondary School
Law Siu-Lung	Diocesan Boys' School
Mok Tze-Tao, Edmond	Queen's College
Poon Wai-Hoi, Bobby	St. Paul's College
Tse Shan-Shan	Tuen Mun Government Secondary School
Yu Chun-Ling	Ying Wa College

38th IMO, July 1997, Mar del Plata, Argentina

Chan Chung-Lam	Bishop Hall Jubilee School
Cheung Pok-Man	STFA Leung Kau Kui College
Lau Lap-Ming, Alvin	St. Paul's College
Leung Wing-Chung	Queen's Elizabeth School
Mok Tze-Tao, Edmond	Queen's College
Yu Ka-Chun	Queen's College

39th IMO, July 1998, Taipei, Taiwan

Chan Kin-Hang	Bishop Hall Jubilee School
Cheung Pok-Man	STFA Leung Kau Kui College
Choi Ming-Cheung	King's College
Lau Lap-Ming, Alvin	St. Paul's College
Law Ka-Ho	Queen's Elizabeth School
Leung Wing-Chung	Queen's Elizabeth School

40th IMO, July 1999, Bucharest, Romania

Chan Ho-Leung	Diocesan Boys' School
Chan Kin-Hang	Bishop Hall Jubilee School
Chan Tsz-Hong	Diocesan Boys' School
Law Ka-Ho	Queen's Elizabeth School
Ng Ka-Wing	STFA Leung Kau Kui College
Wong Chun-Wai	Choi Hung Estate Catholic Secondary School

41st IMO, July 2000, Taejon, Korea

Chan Kin-Hang	Bishop Hall Jubilee School
Fan Wai-Tong	St. Mark’ s School
Law Ka-Ho	Queen’ s Elizabeth School
Ng Ka-Wing	STFA Leung Kau Kui College
Wong Chun-Wai	Choi Hung Estate Catholic Secondary School
Yu Hok-Pun	SKH Bishop Baker Secondary School

42nd IMO, July 2001, Washington D. C., U.S.A.

Chan Kin-Hang	Bishop Hall Jubilee School
Chao Khek-Lun Harold	St. Paul’ s College
Cheng Kei-Tsi, Daniel	La Salle College
Ko Man-Ho	Wah Yan College, Kowloon
Leung Wai-Ying	Queen Elizabeth School
Yu Hok-Pun	SKH Bishop Baker Secondary School